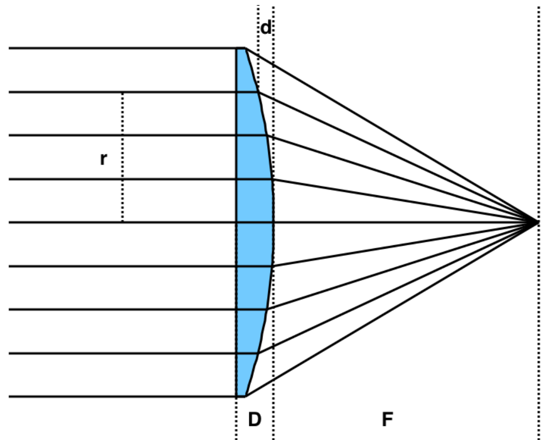
Exam : Waves & Optics

30 January 2017, 9:00-12:00

- Put your name and student number on each answer sheet.
- Answer all questions short and to the point, but complete; write legible.
- Answers that require a unit, but do not have one or the wrong one, are consider incorrect!
- Final point for this exam = total number of 9*points/71 + 1

1. **Lenses (24 points)**
A focussing lens changes the shape of the wavefront from a plane wave to a spherical wave or vice versa. Here, we consider a lens placed in air with a focal length F (defined as the distance from the (center of the) lens to the point at which the spherical wave converges) of 400 mm, a (central) thickness D of 1 cm, with a refractive index of 1.5.



a) How does the phase of an optical wave vary across the wave front of a plane wave? And of a spherical wave? **(2 points)**
A wave front is defined as the plane over which the phase is constant. So the phase doesn't change across the wave front of either. See pg. 26 Hecht

b) What is the relation between the refractive index of a material and the phase velocity of an EM wave? How does the refractive index depend on the electric permittivity and magnetic susceptibility? **(2 points)**
$n = c/v = \sqrt{\varepsilon\mu/\varepsilon_0\mu_0} = \sqrt{K_E K_M}$. See pg. 66 Hecht

c) State Fermat's principle. **(2 points)**
See pg. 106 Hecht: "the actual path between two points taken by a beam of light is the one that is traversed in the least time." Alternatively the modern formulation may be given (see Hecht pg. 109).

d) Give Snell's law of refraction. **(2 points)**
$n_1 \sin\theta_1 = n_2 \sin\theta_2$.

e) Explain briefly what is meant with "dispersion". **(2 points)**
The dependence of the refractive index on the wavelength or frequency of the EM wave.

f) Give the definition of the optical path length. Calculate the optical path length from the entrace surface of the lens to the focal point for a ray going through the center of the lens. **(2 points)**
$OPL = \Lambda = \int_S^P n(s)ds$ (S is source point, *i.e.* point at the entrance surface and P is (in this case) the focal point; alternative: with sum instead of integration, or in words: refractive index weighted (geometric) path length. Calculation: $\Lambda = n \cdot D + F = 1.5 \cdot 1\,\text{cm} + 400\,\text{mm} = 41.5\,\text{cm}$.

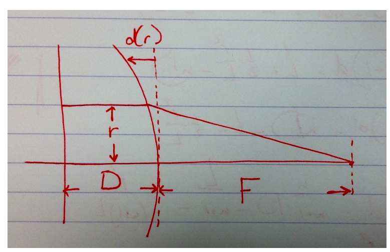
g) Derive that the thickness of the lens can be parametrized as $D - A \cdot r^2$ and calculate A. Assume that the entrance of the lens is flat. Hint: use that $\sqrt{1+2x} \simeq 1 + x$, and ignore terms d^n with $n > 1$. Explain your quantities with a drawing. **(6 points)**
To match/connect the plane wave and the spherical wave, the OPL for each ray must be the same. There are two pieces to be considered: the path *through* the lens, which we call

Λ_1 and the path *behind* the lens (Λ_2). The part *before* the lens could also be considered, but can easily be argued to be the same for all rays (plane wave on flat surface). Both Λ_1 and Λ_2 will depend on the distance of the ray to the axis of the lens (indicated as r in the figure). It is given the thickness of the lens can be parameterized as $T(r) = D - A \cdot r^2 = D - d(r)$ (as indicated/defined in the figure). OPL through lens: $\Lambda_1(r) = n \cdot T(r) = n[D - d(r)]$. OPL from back of lens to focal point: $\Lambda_2(r) = \sqrt{(F+d)^2 + r^2}$. Solve for $\Lambda_1 + \Lambda_2 = C$. Use that $C = nD + F$ to arrive at $d(r) = \frac{1}{2(n-1)}\frac{r^2}{F}$. So $A = \frac{1}{2(n-1)F}$.
In detail:

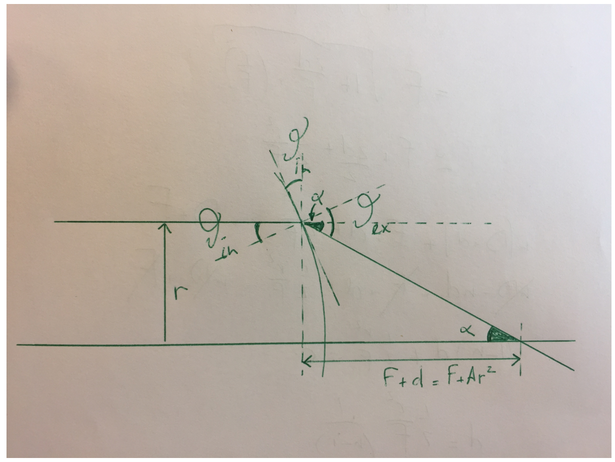
$$\begin{aligned}
nD + F &= n[D-d] + \sqrt{(F+d)^2 + r^2} \\
&= n[D-d] + \sqrt{F^2 + 2dF + d^2 + r^2} \\
&\simeq n[D-d] + \sqrt{F^2 + 2dF + r^2} \quad \text{(use second hint)} \\
&= n[D-d] + F\sqrt{1 + 2d/F + (r/F)^2} \\
&\simeq n[D-d] + F(1 + d/F + (r/F)^2/2) \quad \text{(use first hint)} \\
&= nD - nd + F + d + r^2/2F
\end{aligned}$$

From this is follows that

$$\begin{aligned}
\cancel{nD} + \cancel{F} &= \cancel{nD} - nd + \cancel{F} + d + r^2/2F \\
0 &= -nd + d + r^2/2F \\
(n-1)d &= r^2/2F \\
d &= r^2/2(n-1)F \\
&= 1/2(n-1)F \cdot r^2 \\
&= A \cdot r^2
\end{aligned}$$



h) Use Snell's law to show that a ray that enters the lens 1 cm from the centerline crosses the focal point. If you couldn't solve g), use Snell's law to calculate A. **(4 points)**
Snell's law relates angle of incidence and angle of exit. Needed are the normal on the lens surface (which follows from the previous question), from which then the location at which the ray crosses the centerline can be calculated. Alternatively, assuming the ray crosses the centerline at the focal point, the orientation of the normal on the surface can be calculated. Calculation: Angle of incidence $\theta_{in} \simeq \tan\theta_{in} = \partial D/\partial r = 2Ar$. From $n \sin\theta_{in} = \sin\theta_{ex}$, angle of exit $\theta_{ex} = \sin^{-1}(n\sin\theta_{in}) \simeq n\theta_{in} = 2nAr$. This is w.r.t. the surface, which is tilted by θ_{in} (see also figure), so the angle with respect to the axis is $\alpha = \theta_{ex} - \theta_{in} = n\theta_{in} - \theta_{in} = (n-1)\theta_{in} = 2(n-1)Ar$. For the ray from the focal point to the point of exit, $\alpha \simeq \tan\alpha = \frac{r}{F+Ar^2} \simeq \frac{r}{F}$. Equating both expressions for α yields $2(n-1)Ar = r/F$ or $2(n-1)A = 1/F$ (note that r drops out!). Filling in $A = 1/2(n-1)F$ from the previous questions shows this is indeed true. Alternatively, you find this expression for A assuming the equality holds.



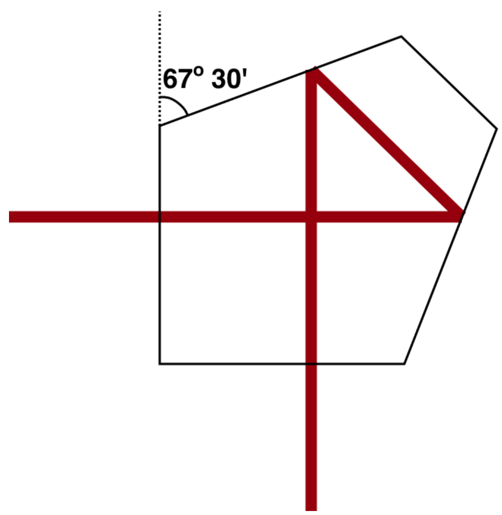
i) If the refractive index for blue light is 1% larger than for green light, will the lens focus the blue light *before* or *behind* the green light? **(2 points)**
For blue light the variation of the OPL as a function of r is larger (the effect of the lens material is 1% larger). Hence the dependence of the path length in the air behind the lens must depend more strongly on r. This happens if the light converges *closer* to the lens than the green light. Alternatively: the curvature of the lens is fixed, hence A is constant, and thus $(n-1)F$. If n goes up, F must go down.

2. **Fiber Communication (21 points)**

a) What is meant by "total internal reflection"? Under what condition does it occur? **(2 points)**

The situation that light reflects 100% from an interface between an optically think medium and an optically thing medium, *i.e.* with $n_{in} > n_{out}$. TIR occurs when for the angle of incidence it holds that $n_{in} \sin\theta_{in}/n_{out} > 1$, since then $\sin\theta_{out} > 1$ should hold. Also: $\sin\theta_{in} > n_{out}/n_{in}$. See Hecht pg. 116.

b) Consider the (symmetric) penta-prism in the figure below. When suspended in air, find out for which refractive indices it would work. **(4 points)**



The angle of incidence on the tilted surfaces is $\theta_{in} = 45°/2 = 22.5°$ and thus $\sin\theta_{in} \simeq 0.38$. This angle has to be equal or larger than the critical angle for which $\sin\theta_c = n_{out}/n_{in}$. So we have $0.38 > \sin\theta_{out} = n_{out}/n_{in} = 1/n_{in}$, so $n_{in} > 2.6$.

c) Give the definitions for the phase and group velocities. Can they be different, and if so, under what condition(s)? **(3 points)**

$v_{phase} = \overline{\omega}/\overline{k}$; $v_{group} = (d\omega/dk)_{\overline{\omega}}$ (see Hecht pg. 296). If there is dispersion, *i.e.* if the refractive index is frequency dependent, group and phase velocity will in general be different.

d) For communication over large distances a train of short light pulses is sent through an optical fiber. Explain how dispersion limits the useful length of the fiber. **(4 points)**

A short pulse contains many frequency components, which follows from Fourier analysis (see Hecht pg. 311). If the corresponding waves propagate at different speeds (= definition of dispersion), the shape will gradually change and typically get wider. At some point, consecutive pulses will overlap and the information is lost.

e) What is a standing wave? How can you create one? **(2 points)**

Standing waves are waves for which the space and time dependence are independent, *i.e.* they are of the form $f(t) \times g(x)$ (see Hecht pg. 288). They appear when two waves of equal frequency counter-propagate and superpose.

f) Light for fiber-communication is generally generated by a laser, which consists of a light amplifying medium placed inside a Fabry-Perot etalon with a very high coefficient of finesse.

Explain why this results in a beam of light with a very large coherence length. **(6 points)**
In a parallel mirror cavity (=Fabry-Perot etalon) only wavelengths for which $L = n\lambda + \delta$ interfere constructively (see Hecht pg. 421). The larger the finesse, the smaller the deviation δ may be. So a high finesse results in a small frequency spread. The latter is inversely proportional to the coherence length (See Hecht pg. 314).

3. **Radio Telescope Array (26 points)**
We will consider the detection of radio waves from a far away galaxy using a set of dish-shaped radio detectors. The figure below shows the Very Large Array (VLA) as an example of such a setup.

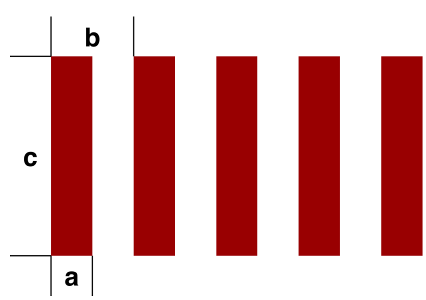
a) Give an equation for the electric field (in 3D) of a plane wave traveling in vacuum with a wavelength λ. **(3 points)**
This needs the relation between wavelength and wavevector $|\vec{k}| = k = 2\pi/\lambda$, and the frequency $\lambda = c/\nu = 2\pi c/\omega$. With those $\vec{E}(t,\vec{r}) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$ or $\vec{E}(t,\vec{r}) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Note that both $\vec{E}$ and $\vec{k}$ must be vector quantities, but that $\vec{k}\cdot\vec{r}$ needs to be a scalar. So $\vec{k}\cdot\vec{r} = 2\pi/\lambda\vec{r}$ is NOT correct!

b) What is the relation between the electric and magnetic field amplitude of an electromagnetic wave (in vacuum)? How are the electric and magnetic field vectors oriented with respect to each other and the wave vector? **(3 points)**
$E = cB$; $\vec{E}$ and $\vec{B}$ are orthogonal, and both orthogonal to $\vec{k}$.

c) What are the definitions of the Poynting vector and irradiance? **(2 points)**
$\vec{S} = c^2\varepsilon_0\vec{E}\times\vec{B}$ (see Hecht pg. 48). Irradiance is the time average of the Poyting vector, $I = \langle S\rangle_T = \frac{1}{2}c\varepsilon_0 E_0^2$, with the averaging period T considerable larger than the period of the oscillation (see Hecht pg. 50).

d) What is the difference between Fraunhofer and Fresnel diffraction? **(2 points)**
Fraunhofer: far-field, distance of source and observer from aperture(s) are large compared to aperture sizes. Fresnel: this is not the case. See Hecht pg. 447.

e) Give the equation for the 2D Fraunhofer interference pattern for the 2D slit configuration shown below (colored areas represent holes in an otherwise opaque screen). Assume the wavelength is λ. **(6 points)**



Horizontal and vertical diffraction are independent. Use $k = 2\pi/\lambda$. The vertical diffraction pattern matches that of a single slit with characteristic parameter $\gamma = kc/2 \sin\theta_V$ (here θ_V is the vertical angle under which the diffraction pattern is observed). The horizontal pattern is a combination of the diffraction pattern for a single slit with width a and thus characteristic scale $\alpha = ka/2 \sin\theta_H$, and that of a multiple-slit configuration with separation b, with characteristic scale $\beta = kb/2 \sin\theta_H$.

Calculation:

$I = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin N\beta}{\sin\beta}\right)^2 \left(\frac{\sin\gamma}{\gamma}\right)^2$, with $k = 2\pi/\lambda$, $\alpha = ka/2 \sin\theta_H$, $\beta = kb/s \sin\theta_H$, and $\gamma = kc/2 \sin\theta_V$. Here $\theta_H \simeq x/L$ and $\theta_V \simeq y/L$, with L the distance from the screen to the slits, x the direction along a and b and y the direction along c. See e.g. Hecht eqn. 10.6, 10.17,10.35.

f) Briefly explain what the Airy disk is. **(2 points)**

The central spot of the diffraction pattern of a circular aperture in Fraunhofer diffraction. See Hecht pg. 469.

g) Argue (don't calculate!) which dimension of the VLA, the dish diameter, the separation between the dishes, or the size of the entire array of dishes, determines the smallest object that can be "seen" (or resolved) by it? **(4 points)**

From the result above, the width of the central peak/spot is determined by the product $N\beta$, which is proportional to the largest dimension of the array. This matches the experience that the largest dimension in spatial dimensions gives rise to the smallest dimension in angular dimension.

h) Will the detection of radio waves from a distant source suffer more or less from Rayleight scattering in the Earth's atmosphere than optical detection? Explain. **(4 points)**

Radio waves will suffer a lot less from Rayleigh scattering. The amount of Rayleigh scattering scales with $1/\lambda^4$. Radio waves have a much longer wavelength than optical waves.